A GEOMETRIC PROOF OF ASPLUND'S DIFFERENTIABILITY THEOREM

BY

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ABSTRACT

A geometric consequence in B of local uniform rotundity in B^* is used to prove Asplund's theorem on Fréchet differentiability of convex functionals.

In 1968 Asplund [1] published a theorem on differentiation of convex funcions.

THEOREM. *If B is a Banach space for which B^* is rotund (locally uniformly rotund), then for each convex open subset E of B and each convex, continuous real-valued function f defined in E, there is a dense G_δ subset D of E such that f has a Gateaux (Fréchet) derivative at each point of D.*

Section 1 contains the definition, after Asplund, modified to avoid the reduction to Lipschitz functions, of a particular dense G_δ subset D of E. Section 2 derives a geometric condition in B from (LUR) in B^* and then show from a simple picture why $a \notin D$ if f is not F-differentiable at a. Section 3 uses a result from my book [2, VII, 2, (10c): B^* is rotund if and only if every two-dimensional factor space of B is smooth] to reduce the problem to dimension two where, for example the result of Section 2 can be applied.

1.

Let $h(x) = \| x \|^2/2$ and define for each $p \geqq 1$ the function $h_p(x) = h(px)/p$. (Since h is quadratic-homogeneous, $h_p(x) = ph(x)$.) Let U be the open unit ball of B.

Given f, take y in E; then take $d > 0$ such that f is bounded on $y + dU$; then use only numbers $p \geqq 1$ and $\lambda \geqq f(y)$ such that

(A) $\quad h_p(x - y) + \lambda - f(x) \geqq 0$ for all x in $y + dU$, $\geqq 2/p$ if $\| x - y \| = d$.

For each such allowed choice of x, d, p, λ, and each positive integer n, define $G(y, d, p, \lambda, n) = \{x: \|x - y\| < d$ and $h_p(x - y) + \lambda - f(x) < 1/pn\}$; let G_n be the union of $G(y, d, p, \lambda, n)$ over all combinations of y, d, p, and λ which satisfy (A); let D be the intersection of the sets G_n.

Each G_n is open. Also by enlarging p ($pd^2/2 > 2/p + \sup\{f(x) - f(y): \|x - y\| < d\}$ will be large enough), λ can then be reduced to drop the graph of $h_p(x - y) + \lambda$ down close to the graph of f at some points of $y + dU$; hence $(G(y, d, p, \lambda, n))$ is an open subset of $y + dU$ which is nonempty when p is large and λ suitably small. This shows that each G_n is dense and open in E, so D is a dense G_δ subset of E.

Some test examples: Let α and β be elements of B^* and let g be the supremum $\alpha \vee \beta$; then d can be arbitrary. In dimension one, it is clear that if h is smooth and $h(x - y) + \lambda - g(x) \geqq 0$ for all x, then there is a positive δ with $h(-y) + \lambda = \delta > 0 = g(0)$, so $0 \notin G_n$ if $n > 1/\delta$. In dimension two, we see that compactness of the unit ball and smoothness of h imply uniform smoothness of h so that no translation by (y, λ) of the smooth graph of h will fit it closer than some positive δ to the graph of g at any point in the line where $\alpha = \beta$. By the similarity of h_p and h, D is the complement of that line.

2.

Assume that B^* is locally uniformly rotund; then we shall show that if f is not Fréchet differentiable at 0, then $0 \notin D$; since the origin may be moved to $(a, f(a))$, this disposes of the general case.

Let the graph of α support that of f at $(0,0)$; since α is not the F-derivative of f at 0, there is an $\varepsilon > 0$ such that for each $\rho > 0$ there is a b with $\|b\| < \rho$ and $f(b) - \alpha(b) > \varepsilon\|b\|$. Choose β, by Hahn-Banach, so that the graph of $\beta - e$ supports the graph of f at $(b, f(b))$; then $\|\alpha - \beta\| \geqq \varepsilon$. (Proof: For all $t \geqq 1$, $\beta(tb) - e \geqq \alpha(tb) + \varepsilon\|tb\|$; divide by t and let t increase to obtain $[\beta - \alpha](b) \geqq \varepsilon\|b\|$.) Let $\phi = \alpha \bigvee (\beta - e)$. Recall the definition of the Fenchel dual of h from [1]: $h^*(\xi) = \sup\{\xi(x) - h(x): x \in B\}$, and dually to compute h from h^*. (Then note that $e = f^*(\beta)$.) Let $\theta = (\alpha - h^*(\alpha)) \bigvee (\beta - h^*(\beta))$ and let Z be the set where $\alpha - h^*(\alpha)$ and $\beta - h^*(\beta)$ are equal; let $\delta = \delta(\alpha, \varepsilon)$ be that positive number which exists by the assumption that h^* is (LUR) at α. One calculation is needed to bring this to a geometric property in B.

(B) *If* $p > 1$ *and* $w \in Z/p$, *then* $h_p(w) \geqq \theta_p(w) + \delta/p$.

PROOF. If $z \in Z$, then

$$h(z) = \sup\{\xi(z) - h^*(\xi) : \xi \in B^*\} \geqq [\alpha + \beta](z)/2 - h^*((\alpha + \beta)/2)$$
$$\geqq (\alpha(z) + \beta(z))/2 - (h^*(\alpha) + h^*(\beta))/2 + \delta$$
$$= (\alpha(z) - h^*(\alpha))/2 + (\beta(z) - h^*(\beta))/2 + \delta = \theta(z) + \delta.$$

(B) follows by the similarity of the graphs of θ_p and h_p to those of θ and h.

Let c be the point between 0 and b where $\alpha(c) = \beta(c) - e$. Now for any choice of y, d, p, and λ which satisfies (A) and for which $\| y - c \| < d$ and $pd > \| \alpha \| \vee \| \beta \|$, we have $h_p(x - y) + \lambda \geqq \phi(x)$ for all x, so $\theta_p(x - y) \geqq \phi(x)$ for all x. Then, following parallels to the graph of α or β in Fig. 1, and using (B), we have $h_p(c - y) + \lambda - \alpha(c) \geqq h_p(w) - \theta_p(w) \geqq \delta/p$.

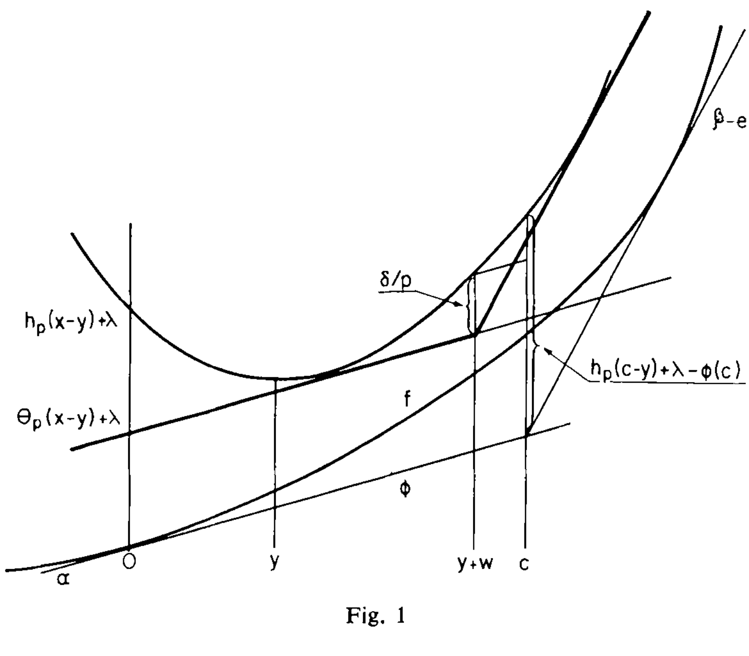

Fig. 1

Since a convex, continuous f is locally Lipschitz, p can be taken large enough, simultaneously for all b and hence c, near 0. For such a p, h_p and α are continuous so $h_p(-y) + \lambda - f(0) = h_p(-y) + \lambda - \alpha(0) \geqq \delta/p$. Hence $0 \notin G_n(y, d, p, \lambda, n)$ if $n \geqq 1/\delta$ and p is large. It follows that $0 \notin G_n$ for large n, so $0 \notin D$.

3.

Assume that B^* is rotund; we show that if f is not G-differentiable at 0, then $0 \notin D$. Move the origin as before to $(a, f(a))$ and assume α supports f at $(0,0)$.

For each closed linear subspace H of $\alpha_\perp$, define $B' = B/H$, $x = x' + H$, $f'(x') = \inf\{f(x): x \in x'\}$, $h'(x') = \inf\{h(x): x \in x'\}$, and so on. Then:

(a) E' is convex and open and f' is convex and continuous on it.

(b) $h'(x') = \| x' \|^2/2$.

(c) The graph of α' supports that of f' at $(0', 0)$.

(d) If $0 \in G(y, d, p, \lambda, n)$, then y, d, p, and λ satisfy (A) for h' and f', and $f(0) = 0 = f'(0')$, so $0' \in G'(y', d, p, \lambda, n)$.

(e) If $0 \in D$, then $0' \in D'$.

But if α is not the G-derivative of f at 0, then there is also a $\beta \neq \alpha$ whose graph supports that of f at $(0,0)$; let $H = \alpha_\perp \cap \beta_\perp$. Then α' and β' both support f' at $(0', 0)$. By [2, Chap. VII, Section 2, (10c)], B' is smooth; because it is two-dimensional, B' is uniformly smooth, so Section 2 says that $0' \notin D'$. By (e) above, $0 \notin D$.

REMARKS. (1) Asplund [1] calls a space, in which the conclusion of the theorem holds, a weak (or strong) differentiability space. He notes that because the conclusion of his theorem is independent of the specific norm chosen from its isomorphism class, B is a strong differentiability space if B has an isomorphic norm whose conjugate norm is (LUR); for example it suffices that B^* be separable [1] or that B be reflexive [3].

(2) This new proof does not settle whether the condition that B^* be (LUR) is really necessary. For example, it is not known whether F-(G-)differentiability everywhere of the norm in B is either necessary or sufficient for B to be a strong (weak) differentiability space. It seems, however, that an f and a G-differentiable norm can be given in c_0 so that $0 \in D$ but f is not G-differentiable at 0. This, if it works out, would show that this method of proof, like Asplund's, is too closely tied to the specific norm to settle the question above.

REFERENCES

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